

# NO COHOMOLOGICALLY TRIVIAL NON-TRIVIAL AUTOMORPHISM OF GENERALIZED KUMMER MANIFOLDS

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**ABSTRACT.** We prove that the natural representation, on the total Betti cohomology group, of the automorphism group of a hyperkähler manifold deformation equivalent to a generalized Kummer manifold is faithful. This is a sort of generalization of an earlier work of Beauville and a recent work of Boissière, Nieper-Wisskirchen and Sarti, concerning the representation on the second cohomology group.

## 1. INTRODUCTION

Our main result is Theorem (1.3). Throughout this note, we work over  $\mathbf{C}$ .

Global Torelli theorem for K3 surfaces ([PS71], [BR75], see also [BHPV04]) says that the natural contravariant representation  $\rho_2 : \text{Aut}(S) \rightarrow \text{GL}(H^2(S, \mathbf{Z}))$  is faithful for any K3 surface  $S$ . On the other hand, Dolgachev ([Do84, 4.4]) and Mukai and Namikawa ([MN84], [Mu10]) show that there are Enriques surfaces  $E$  such that  $\rho_2 : \text{Aut}(E) \rightarrow \text{GL}(H^2(E, \mathbf{Z}))$  are not faithful. Here and hereafter  $\text{GL}(L) := \text{Aut}_{\text{group}}(L)$  for a finitely generated abelian group  $L$ , possibly with non-trivial torsion.

Beauville ([Be83-2, Propositions 9, 10]) considered a similar question for hyperkähler manifolds, by which we mean a simply-connected compact Kähler manifold  $M$  admitting everywhere non-degenerate global 2-form  $\omega_M$  such that  $H^0(M, \Omega_M^2) = \mathbf{C}\omega_M$ . Most standard examples are the Hilbert scheme  $\text{Hilb}^n(S)$  of dimension  $2n$  of 0-dimensional closed subschemes of length  $n$  on a K3 surface  $S$ , the generalized Kummer manifold  $K_{n-1}(A)$  of dimension  $2(n-1) \geq 4$  associated to a 2-dimensional complex torus  $A$  (see eg. Section 2 for definition) and their deformations ([Be83, Sections 6,7]). For them, he found the following two opposing phenomena:

**Theorem 1.1.** (1)  $\rho_2 : \text{Aut}(\text{Hilb}^n(S)) \rightarrow \text{GL}(H^2(\text{Hilb}^n(S), \mathbf{Z}))$  is faithful.  
(2)  $\rho_2 : \text{Aut}(K_{n-1}(A)) \rightarrow \text{GL}(H^2(K_{n-1}(A), \mathbf{Z}))$  is not faithful. More precisely,  $T(n) \subset \text{Ker } \rho_2$ . Here  $T(n) \simeq (\mathbf{Z}/n)^{\oplus 4}$  is the group of automorphisms naturally induced from the group of  $n$ -torsion points  $T[n] := \{a \in A \mid na = 0\}$  of  $A = \text{Aut}^0(A)$ .

Recently, Boissière, Nieper-Wisskirchen and Sarti ([BNS11, Theorem 3, Corollary 5 (2)]) generalized the second statement completely as follows:

**Theorem 1.2.**  $\text{Ker}(\rho_2 : \text{Aut}(K_{n-1}(A)) \rightarrow \text{GL}(H^2(K_{n-1}(A), \mathbf{Z}))) = T(n) \cdot \langle \iota \rangle$ . Here  $\iota$  is the automorphism naturally induced from the inversion  $-1$  of  $A$  and  $\cdot$  is the semi-direct product for which  $T(n)$  is normal.

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Now, a natural question is how about the representation on the total cohomology group  $H^*(K_{n-1}(A), \mathbf{Z}) := \bigoplus_{k=0}^{4(n-1)} H^k(K_{n-1}(A), \mathbf{Z})$ . Our main result is the following:

**Theorem 1.3.** *Let  $Y$  be any hyperkähler manifold deformation equivalent to  $K_{n-1}(A)$ . Then, the natural representation  $\rho : \text{Aut}(Y) \rightarrow \text{GL}(H^*(Y, \mathbf{Z}))$  is faithful.*

First we prove Theorem (1.3) for  $K_{n-1}(A)$ . By Theorem (1.2), it suffices to show that  $g^*|_{H^*(K_{n-1}(A), \mathbf{Z})} \neq \text{id}$  for each  $g \in T(n) \cdot \langle \iota \rangle \setminus \{\text{id}\}$ . This will be done in Sections 3. We then prove Theorem (1.3) for any  $Y$  by using the density result due to Markman and Mehrotra ([MM12, Theorem 4.1, see also Theorem 1.1]) in Section 4. In Remark (4.2), we point out a similar result for deformation of  $\text{Hilb}^n(S)$ . After posting this note on ArXiv, Professor Yuri Tschinkel kindly informed that in his joint work with Hassett ([HT10, Theorem 2.1, Proposition 3.1]), they show that the natural action  $T(n) \cdot \langle \iota \rangle$  on  $K_{n-1}(A)$  above extends to a faithful action on any deformation  $Y$  of  $K_{n-1}(A)$ , being trivial on  $H^2(Y, \mathbf{Z})$ . In particular, we have  $\text{Aut}(Y) \neq \{\text{id}_Y\}$  in Theorem (1.3) even if  $Y$  is generic.

Besides its own theoretical interest, Theorem (1.3) is much inspired by the following question asked by Professor Dusa McDuff to me at the conference in Banff (2012, July):

**Question 1.4.** Is there an example of a compact Kähler manifold  $M$  such that the biholomorphic automorphism group is discrete, i.e.,  $\text{Aut}^0(M) = \{\text{id}_M\}$ , but with a biholomorphic automorphism  $g \neq \text{id}_M$  being homotopic to  $\text{id}_M$  in the group of diffeomorphisms?

Note that any translation automorphism of a complex torus  $T$  of  $\dim T > 0$  is holomorphically homotopic to  $\text{id}_T$ , but  $\text{Aut}^0(T) = T$ . If such  $g \in \text{Aut}(M)$  exists, then  $g$  is necessarily cohomologically trivial, i.e.,  $g \in \text{Ker}(\rho : \text{Aut}(M) \rightarrow \text{GL}(H^*(M, \mathbf{Z})))$ .

Cohomologically trivial automorphisms on Enriques surfaces are not affirmative examples of Question (1.4). In fact, any automorphism  $g$  on an Enriques surface lifts to automorphisms  $\tilde{g}$  of the covering K3 surface in two ways. As pointed out by her, one of the lifted automorphisms is homotopic to  $\text{id}$  if so is the original one, in the groups of diffeomorphisms. But none of the two lifted automorphisms is cohomologically trivial unless  $\tilde{g} = \text{id}$  by the global Torelli theorem for K3 surfaces. As other candidates, I wonder if Beauville's automorphisms  $T(n)$  of  $K_{n-1}(A)$  in Theorem (1.1) provide affirmative examples or not, as it is derived from translations of  $A$ . This is my starting point of this work. Unfortunately, Theorem (1.3) says that they are not even cohomologically trivial. So, *Question (1.4) itself is still completely open and also widely open even for cohomologically trivial level.*

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## 2. PRELIMINARIES.

In this section, following [Be83, Section 7], we mainly fix notations on a generalized Kummer manifold. For more details on generalized Kummer manifolds and basic properties on hyperkähler manifolds, see [Be83, Section 7] and [GHJ03, Part III]. In order to avoid any notational confusion, we also notice here that  $K_n(A)$  in [BNS11] is  $K_{n-1}(A)$  in this note and [Be83], [Be83-2].

Let  $A$  be a 2-dimensional complex torus and let  $n$  be an integer such that  $n \geq 3$ . Let  $\text{Hilb}^n(A)$  be the Hilbert scheme of 0-dimensional closed subschemes of  $A$  of length  $n$ . Then  $\text{Hilb}^n(A)$  is a smooth Kähler manifold of dimension  $2n$ . Let

$$\nu = \nu_A : \text{Hilb}^n(A) \rightarrow \text{Sym}^n(A) = A^n/S_n$$

be the Hilbert-Chow morphism. We denote the sum as 0-cycles by  $\oplus$  and the sum in  $A$  by  $+$ . Then an element of  $\text{Sym}^n(A)$  is of the form

$$\oplus_{i=1}^k x_i^{\oplus m_i} ,$$

where  $x_i$  are distinct points on  $A$  and  $m_i$  are positive integers such that  $\sum_{i=1}^k m_i = n$ . Then, we have a natural surjective morphism

$$s := s_A : \text{Sym}^n(A) \rightarrow A ; \oplus_{i=1}^k x_i^{\oplus m_i} \mapsto \sum_{i=1}^k m_i x_i .$$

Hence we have a surjective morphism

$$s \circ \nu = s_A \circ \nu_A : \text{Hilb}^n(A) \rightarrow A .$$

This is a smooth morphism such that all fibers are isomorphic. Any fiber of  $s_A \circ \nu_A$  is the generalized Kummer manifold  $K_{n-1}(A)$ . This is in fact a  $2(n-1)$ -dimensional hyperkähler manifold. When  $n = 1$ , it is a point and when  $n = 2$ , it is the Kummer K3 surface  $\text{Km}(A)$  associated to  $A$ . So, we usually consider only the cases where  $2(n-1) \geq 4$ , i.e.,  $n \geq 3$ .

We fix  $K_{n-1}(A) = (s \circ \nu)^{-1}(0)$ , the fiber over 0 from now on until the end of this note. We can also describe  $K_{n-1}(A)$  in a slightly different way. Let

$$A(n-1) := \{(P_1, P_2, \dots, P_n) \in A^n \mid \sum_{i=1}^n P_i = 0\} .$$

$A(n-1)$  is a closed submanifold of  $A^n$ , isomorphic to  $A^{n-1}$ .  $A(n-1)$  is stable under the action of  $S_n$  on  $A^n$ , and we have:

$$\text{Sym}^n(A) \supset A^{(n-1)} := s^{-1}(0) = A(n-1)/S_n .$$

Our  $K_{n-1}(A)$  is then:

$$K_{n-1}(A) = \nu^{-1}(A^{(n-1)}) = \text{Hilb}^n(A) \times_{\text{Sym}^n(A)} A^{(n-1)} .$$

It is known that  $\dim \text{Def}(A) = 4$ , while  $\dim \text{Def}(K_{n-1}(A)) = 5$  for  $n \geq 3$  and any local deformation of hyperkähler manifold is again a hyperkähler manifold. So, there are hyperkähler manifolds  $Y$  which are deformation equivalent to  $K_{n-1}(A)$  but not isomorphic to any generalized Kummer manifold.

### 3. PROOF OF THEOREM (1.3) FOR $K_{n-1}(A)$ .

From now until the end of this note, we put  $X := K_{n-1}(A)$  and  $K := T(n) \cdot \langle \iota \rangle$  (see Introduction for definition), and employ the same notations in Section 2.

In this section, we prove Theorem (1.3) for  $K_{n-1}(A)$  with  $2(n-1) \geq 4$ . By Theorem (1.2), it suffices to prove the following two Claims:

**Claim 3.1.** Let  $g \in K \setminus T(n)$ . Then  $g^*|H^*(X, \mathbf{C}) \neq \text{id}$ .

**Claim 3.2.** Let  $a \in T(n) \setminus \{id\}$ . Then  $a^*|H^*(X, \mathbf{C}) \neq id$ .

Let  $(z_i^1, z_i^2)$  ( $1 \leq i \leq n$ ) be the standard global coordinates of the universal cover  $\mathbf{C}_i^2$  of the  $i$ -th factor  $A_i = A$  of the product  $A^n$ . Then the universal cover  $\mathbf{C}^{2(n-1)}$  of  $A(n-1) \simeq A^{n-1}$  is a closed submanifold of  $\mathbf{C}^{2n}$ , defined by the two equations:

$$\begin{aligned} z_1^1 + z_2^1 + \cdots + z_{n-1}^1 + z_n^1 &= 0, \\ z_1^2 + z_2^2 + \cdots + z_{n-1}^2 + z_n^2 &= 0. \end{aligned}$$

In particular,  $(z_i^1, z_i^2)$  ( $1 \leq i \leq n-1$ ) give the global linear coordinates of the universal cover  $\mathbf{C}^{2(n-1)}$  of  $A(n-1)$ . Note that  $dz_i^1$  and  $dz_i^2$  ( $1 \leq i \leq n$ ) can be regarded as global 1-forms on  $A(n-1)$ . They satisfy

$$\begin{aligned} dz_1^1 + dz_2^1 + \cdots + dz_{n-1}^1 + dz_n^1 &= 0, \\ dz_1^2 + dz_2^2 + \cdots + dz_{n-1}^2 + dz_n^2 &= 0. \end{aligned}$$

So,  $dz_i^1$  and  $dz_i^2$  ( $1 \leq i \leq n-1$ ) form the basis of the space of global holomorphic 1-forms on  $A(n-1) \simeq A^{n-1}$ .

**Lemma 3.3.** Consider the following global  $(2,1)$ -form  $\tilde{\tau}$  on  $A(n-1)$ :

$$\tilde{\tau} = dz_1^1 \wedge dz_1^2 \wedge d\bar{z}_1^2 + dz_2^1 \wedge dz_2^2 \wedge d\bar{z}_2^2 + \cdots + dz_{n-1}^1 \wedge dz_{n-1}^2 \wedge d\bar{z}_{n-1}^2 + dz_n^1 \wedge dz_n^2 \wedge d\bar{z}_n^2.$$

Then  $\tilde{\tau}$  naturally descends to a non-zero element  $\tau$  of  $H^{2,1}(X)$ .

*Proof.* Since  $\tilde{\tau}$  is  $S_n$ -invariant, it descends to a global  $(2,1)$ -form, say  $\bar{\tau}$ , on the compact Kähler V-manifold  $A^{(n-1)}$ . For compact Kähler V-manifolds, the Hodge decomposition is pure and the Hodge theory works in the same way as smooth compact case ([St77]). Consider  $\nu|X : X \rightarrow A^{(n-1)}$ . Then  $\tau = (\nu|X)^*\bar{\tau} \in H^{2,1}(X)$ . It remains to show that  $\tau \neq 0$ . Since  $(\nu|X)^*$  is injective, it suffices to show that  $\bar{\tau} \neq 0$  in  $H^{2,1}(A^{(n-1)})$ . For this, it suffices to show that  $\tilde{\tau} \neq 0$  in  $H^{2,1}(A(n-1))$ , again by the injectivity applied for the quotient map  $A(n-1) \rightarrow A^{(n-1)}$ . By the two equalities above, we have

$$\tilde{\tau} = dz_1^1 \wedge dz_1^2 \wedge d\bar{z}_1^2 + dz_2^1 \wedge dz_2^2 \wedge d\bar{z}_2^2 + \cdots + dz_{n-1}^1 \wedge dz_{n-1}^2 \wedge d\bar{z}_{n-1}^2 - \left( \sum_{k=1}^{n-1} dz_k^1 \right) \wedge \left( \sum_{k=1}^{n-1} dz_k^2 \right) \wedge \left( \sum_{k=1}^{n-1} d\bar{z}_k^2 \right).$$

This gives an expression of  $\tilde{\tau}$  in terms of the standard basis of  $H^{2,1}(A(n-1))$ . By  $n-1 \geq 2$ , the term  $dz_1^1 \wedge dz_2^2 \wedge d\bar{z}_2^2$  appears with coefficient  $-1$  and we are done.  $\square$

**Lemma 3.4.** Let  $g \in K \setminus T(n)$  and  $\tau \in H^{2,1}(X)$  be as in Lemma (3.3). Then  $g^*\tau = -\tau$ . In particular,  $g$  is not cohomologically trivial.

*Proof.*  $g$  acts equivariantly for  $A(n-1) \rightarrow A^{(n-1)} \leftarrow X$ . Since  $g \in K \setminus T(n)$ , it follows that  $g^*dz_i^q = -dz_i^q$  ( $1 \leq i \leq n$ ,  $q = 1, 2$ ). Hence  $g^*\tilde{\tau} = -\tilde{\tau}$  by the shape of  $\tilde{\tau}$ . Thus  $g^*\tau = -\tau$ . Since  $\tau \neq 0$  in  $H^{2,1}(X)$ , this implies  $g^*|H^3(X, \mathbf{C}) \neq id$ .  $\square$

Lemma (3.4) completes the proof of Claim (3.1).

From now, we prove Claim (3.2).

The next proposition is a formal generalization of a result of Camere ([Ca10, Proposition 3]), which states and proves for symplectic involutions. Logically, this proposition is not needed in our proof but this helps to guess the result of Proposition (3.6) and may be applicable for other studies. So we state and prove here:

**Proposition 3.5.** *Let  $(M, \omega_M)$  be a holomorphic symplectic manifold of dimension  $2m$ , i.e.,  $M$  is a compact Kähler manifold and  $\omega_M$  is an everywhere non-degenerate holomorphic 2-form on  $M$ . Let  $h \in \text{Aut}(M)$  such that  $h^*\omega_M = \omega_M$  and  $h$  is of finite order, say  $m$ . Let  $F$  be a connected component of the fixed locus  $M^h = \{P \in M | h(P) = P\}$ . Then  $(F, \omega_M|_F)$  is a holomorphic symplectic manifold, in particular, of even dimension. Here we also consider that a point is a symplectic manifold, as it is harmless at all.*

*Proof.*  $F$  is isomorphic to the intersection of the graph of  $h$  and the diagonal in  $M \times M$ . So it is compact and Kähler. Let  $P \in F$ . Since  $h$  is of finite order,  $h$  is locally linearizable at  $P$ . That is, there are local coordinates  $(y_1, y_2, \dots, y_{2m})$  such that  $h^*y_i = y_i$  for all  $1 \leq i \leq r$  and  $h^*y_j = c_j y_j$  for all  $r+1 \leq j \leq 2m$ . Here  $c_j \neq 1$  and satisfies  $c_j^m = 1$ . Then  $F$  is locally defined by  $x_j = 0$  ( $r+1 \leq i \leq 2d$ ), in  $M$ . Hence  $F$  is smooth. Since  $h|_V = \text{id}_V$ , the automorphism  $h$  induces the linear automorphism  $h_{*,P} : T_P M \rightarrow T_P M$  of the tangent space of  $M$  at  $P$ . By the description above, we have the decomposition  $T_P M = T_P V \oplus N$  where  $N = \bigoplus_{j=r+1}^{2m} \mathbf{C} v_j$  with  $h_{*,P}(v_j) = c_j^{-1} v_j$ , and the tangent space  $T_P F$  of  $F$  is exactly the invariant subspace  $(T_P M)^{h_{*,P}}$ . Then for  $v \in T_P F$  and  $v_j$ , we have by  $h^*\omega_M = \omega_M$ :

$$\omega_{M,P}(v, v_j) = (h_P^*(\omega_{M,P}))(h_{*,P}(v), h_{*,P}(v_j)) = \omega_{M,P}(v, c_j^{-1} v_j) = c_j^{-1} \omega_{M,P}(v, v_j) .$$

Since  $c_j \neq 1$ , it follows that  $\omega_{M,P}(v, v_j) = 0$  for all  $r+1 \leq j \leq 2m$ , that is, the decomposition  $T_P M = T_P V \oplus N$  is orthogonal with respect to  $\omega_{M,P}$ . Hence  $\omega_{M,P} = \omega_{M,P}|_{T_P V} \oplus \omega_{M,P}|_N$ . Since  $\omega_{M,P}$  is non-degenerate, it follows that  $\omega_{M,P}|_{T_P V}$  (indeed, also  $\omega_{M,P}|_N$ ) is non-degenerate, and in particular  $\dim F = r$  is even (possibly 0). This completes the proof.  $\square$

From now on, let  $a \in T(n) \simeq (\mathbf{Z}/n)^{\oplus 4}$  be an element of order  $p \neq 1$  ( $p$  is *not* assumed to be prime). Set  $d = n/p$ . Then  $d$  is a positive integer such that  $d < n$ . We freely regard  $a$  also as a torsion element of order  $p$  in  $A$  and automorphisms of various spaces which are naturally and equivariantly induced from the translation automorphism  $x \mapsto x + a$  on  $A$ .

**Proposition 3.6.** *The fixed locus  $X^a$  consists of  $p^3$  connected components  $F_i$  ( $1 \leq i \leq p^3$ ), each of which is isomorphic to the generalized Kummer manifold  $K_{d-1}(A/\langle a \rangle)$  associated to the 2-dimensional complex torus  $A/\langle a \rangle$ .*

*Proof.* Let  $\mathcal{S} \subset A$  be a 0-dimensional closed subscheme of length  $n$ . Since  $\langle a \rangle$  acts freely on  $A$ , or in other words, the quotient map  $\pi : A \rightarrow A/\langle a \rangle$  is étale of degree  $p$ , it follows that  $a_*\mathcal{S} = \mathcal{S}$  if and only if there is a 0-dimensional closed subscheme  $\mathcal{T} \subset A/\langle a \rangle$  of length  $d = n/p$  such that  $\mathcal{S} = \pi^*\mathcal{T}$ . This  $\mathcal{T}$  is unique. Thus we have an isomorphism  $\text{Hilb}^d(A/\langle a \rangle) \simeq (\text{Hilb}^n(A))^a$  by  $\pi^*$ .

Let  $\mathcal{S} \in (\text{Hilb}^n(A))^a$ . Then, by  $a_*\mathcal{S} = \mathcal{S}$ , it follows that  $\nu(\mathcal{S}) \in (\text{Sym}^n(A))^a$  and it is of the form:

$$\nu(\mathcal{S}) = \bigoplus_{i=1}^k \bigoplus_{j=0}^{p-1} (x_i + ja)^{\oplus m_i} ,$$

where  $\sum_{i=1}^k m_i = d$  and all points  $x_i + ja$  are distinct. Note that  $K_{n-1}(A)^a = (\text{Hilb}^n(A))^a \cap K_{n-1}(A)$ . Hence, by definition of  $K_{n-1}(A)$ , we have  $\mathcal{S} \in K_{n-1}(A)^a$  if and only if  $\mathcal{S} \in K_{n-1}(A)$ , whence, if and only if

$$p(m_1 x_1 + m_2 x_2 + \dots + m_k x_k + \alpha) = 0 \quad \text{---} \quad (*)$$

in  $A$ . Here  $\alpha \in A$  is an element such that  $p\alpha = (n(p-1)/2)a$  in  $A$ . (We choose and fix such  $\alpha$ . We also note that  $n(p-1)/2 \in \mathbf{Z}$ .) The last equation (\*) is equivalent to

$$m_1x_1 + m_2x_2 + \cdots + m_kx_k + \alpha \in A[p] \quad \text{---} \text{---} \text{---} (**),$$

where  $A[p]$  is the group of  $p$ -torsion points of  $A$ . Since  $a$  is also a  $p$ -torsion point, this condition (\*\*) is also equivalent to

$$m_1\pi(x_1) + m_2\pi(x_2) + \cdots + m_k\pi(x_k) + \pi(\alpha) \in \pi(A[p]) = A[p]/\langle a \rangle \quad \text{---} \text{---} \text{---} (***) .$$

Write  $\mathcal{S} = \pi^*(\mathcal{T})$ . Then, the last condition (\*\*\*) holds if and only if  $\mathcal{T}$  is in the fibers of

$$s_{A/\langle a \rangle} \circ \nu_{A/\langle a \rangle} : \text{Hilb}^d(A/\langle a \rangle) \rightarrow A/\langle a \rangle$$

over  $\pi(A[p])$ . We have  $|\pi(A[p])| = p^3$ , as  $a$  is also  $p$ -torsion. Hence, by  $\pi^*$ , the fixed locus  $K_{n-1}(A)^a$  is isomorphic to the union of  $p^3$  fibers of  $s_{A/\langle a \rangle} \circ \nu_{A/\langle a \rangle}$ , each of which is, by definition, isomorphic to  $K_{d-1}(A/\langle a \rangle)$ . This completes the proof.  $\square$

We recall the following fundamental result due to Göttsche and Soergel ([GS93, Corollary 1], see also [Go94], [De10]):

**Theorem 3.7.** *The topological Euler number  $\chi_{\text{top}}(K_{n-1}(A))$  of  $K_{n-1}(A)$  is  $n^3\sigma(n)$ , where  $\sigma(n) = \sum_{1 \leq b|n} b$ , the sum of all positive divisors of  $n$ . (This is valid also for  $n = 1, 2$ .)*

Consider now the Lefschetz number of  $h \in \text{Aut}(X)$ :

$$L(h) := \sum_{k=0}^{4(n-1)} (-1)^k \text{tr}(h^*|H^k(X, \mathbf{C})) .$$

**Proposition 3.8.** (1) *If  $h \in \text{Aut}(X)$  is cohomologically trivial, then  $L(h) = n^3\sigma(n)$ .*

(2)  *$L(a) = n^3\sigma(d)$ , where  $d = n/p$ .*

*Proof.* If  $h$  is cohomologically trivial, then  $\text{tr}(h^*|H^k(X, \mathbf{C})) = b_k(X)$ . This implies (1). By the topological Lefschetz fixed point formula, Proposition (3.6) and Theorem (3.7), we have

$$L(a) = \chi_{\text{top}}(X^a) = p^3 \chi_{\text{top}}(K_{d-1}(A/\langle a \rangle)) = p^3 \cdot d^3 \sigma(d) = n^3 \sigma(d) .$$

This is nothing but the assertion (2).  $\square$

Since  $d|n$  and  $d \neq n$ , it follows that  $\sigma(d) \leq \sigma(n) - n < \sigma(n)$ . Hence by Proposition (3.8)(1), (2),  $a$  is not cohomologically trivial. This shows Claim (3.2). Now, the proof of Theorem (1.3) for  $K_{n-1}(A)$  is completed.

#### 4. PROOF OF THEOREM (1.3).

In this section, we shall prove Theorem (1.3).

Let  $\Lambda = (\Lambda, (*, **))$  be a fixed abstract lattice isometric to  $(H^2(K_{n-1}(A), \mathbf{Z}), b)$ ,  $b$  being the Beauville-Bogomolov form. Let  $Y$  be a hyperkähler manifold deformation equivalent to a generalized Kummer manifold  $X = K_{n-1}(A)$ . Let  $g \in \text{Aut}(Y)$  such that  $g^*|H^*(Y, \mathbf{Z}) = \text{id}$ . We shall show that  $g = \text{id}_Y$ .

Let  $\mathcal{M}^0$  be the connected component of the marked moduli space of  $\mathcal{M}_\Lambda$ , containing  $(Y, \eta)$ . Here  $\eta : H^2(Y, \mathbf{Z}) \rightarrow \Lambda$  is a marking. The marked moduli space  $\mathcal{M}_\Lambda$  is constructed by Huybrechts ([Hu99, 1.18]) by patching Kuranishi spaces via local Torelli. By construction,

$\mathcal{M}_\Lambda$  is smooth, but highly non-Hausdorff as explained briefly below. He also showed that the period map

$$p : \mathcal{M}^0 \rightarrow \mathcal{D} = \{[\omega] \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}$$

is a surjective holomorphic map of degree 1 ([Hu99, Theorem 8.1], see also [Ve09], [Hu11] for degree and further development). For  $[\omega] \in \mathcal{D}$ , if  $p^{-1}([\omega]) \subset \mathcal{M}^0$  is not a single point, then  $p^{-1}([\omega])$  consists of points, being mutually inseparable, corresponding to birational hyperkähler manifolds. Markman and Mehrotra ([MM12, Theorem 4.1]) show that marked generalized Kummer manifolds are dense in  $\mathcal{M}^0$ . Actually they proved a strong density (see also an argument of Theorem 1.1 in [MM12]), based on the Hodge theoretic Torelli type Theorem ([Ma11]):

**Theorem 4.1.** *There is a dense subset  $\mathcal{D}' \subset \mathcal{D}$  such that each point of  $p^{-1}([\omega])$  corresponds to a generalized Kummer manifold for  $[\omega] \in \mathcal{D}'$ .*

Consider the Kuranishi family  $u : \mathcal{U} \rightarrow \mathcal{K}$  of  $Y$ . Here and hereafter we freely shrink  $\mathcal{K}$  around  $0 = [Y]$ . Since the Kuranishi family is universal,  $g \in \text{Aut}(Y)$  induces automorphisms  $\tilde{g} \in \text{Aut}(\mathcal{U})$  and  $\bar{g} \in \text{Aut}(\mathcal{K})$  such that  $u \circ \tilde{g} = \bar{g} \circ u$  and  $\tilde{g}|_Y = g$ . Since  $\mathcal{K}$  is locally isomorphic to  $\mathcal{D}$  by the local Torelli theorem ([Be83, Theorem 5]), the locus  $\mathcal{K}' \subset \mathcal{K}$ , consisting of the point  $t$  such that  $u^{-1}(t)$  is a generalized Kummer manifold, is dense in  $\mathcal{K}$ . This is a direct consequence of Theorem (4.1) and the construction of  $\mathcal{M}^0$  briefly explained above. Here we note that the density in  $\mathcal{M}^0$  is not sufficient. Now we follow Beauville's argument ([Be83-2, Proof of Proposition 10]) to conclude.

We can take  $\mathcal{K}$  as a small polydisk in  $H^1(Y, TY)$  with center 0. Recall that  $H^1(Y, TY) \simeq H^1(X, \Omega_Y^1)$  by  $\omega_Y$ . Then  $\bar{g} = id_{\mathcal{K}}$ . Indeed, since  $g$  is cohomologically trivial and  $H^{2,0}(Y) = H^0(Y, \Omega_Y^2) = \mathbf{C}\omega_Y$ , we have  $g^*\omega_Y = \omega_Y$  and  $g^*[H^1(X, \Omega_Y^1)] = id$ . Let  $t \in \mathcal{K}$  be any point of  $\mathcal{K}$ . Then  $\tilde{g}$  preserves the fiber  $Y_t = u^{-1}(t)$ , i.e.,  $\tilde{g}|_{Y_t} \in \text{Aut}(Y_t)$ . Put  $g_t := \tilde{g}|_{Y_t}$ . Then  $g_t$  is also cohomologically trivial, because  $g_t^*$  is derived from the action of  $\tilde{g}$  on the constant system  $\bigoplus_{k=0}^{4(n-1)} R^k u_* \mathbf{Z}$ . But then  $g_t = id_{Y_t}$  over the dense subset  $\mathcal{K}'$  by Theorem (1.3) for  $K_{n-1}(A)$ . Since  $\mathcal{U}$  is Hausdorff and  $\tilde{g}$  is continuous, this implies that  $\tilde{g} = id_{\mathcal{U}}$ . Hence  $g = g_0 = id_Y$  as well. This completes the proof of Theorem (1.3).

**Remark 4.2.** Markman and Mehrotra ([MM12, Theorem 1.1]) also show the strong density result for  $\text{Hilb}^n(S)$  of K3 surfaces  $S$ . So, the same argument here together with Beauville's result (Theorem (1.1)(1)) implies the following: Let  $W$  be a hyperkähler manifold deformation equivalent to  $\text{Hilb}^n(S)$ . Let  $g \in \text{Aut}(W)$  such that  $g^*[H^2(W, \mathbf{Z})] = id$ . Then  $g = id_W$ .

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